RLC circuits and Resonance

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Group lab report

Choose a group member to be the record keeper. (This person should be a different person from the record keeper last week.) After creating the lab report from the template below, make sure to click the "Share" button in the upper right corner and add all group member email addresses.

Using Jupyter notebook

In order to do data processing and analysis, we will use Jupyter notebooks (which run on the Python programming language). Download the following notebook file:

Introduction

In this experiment we wish to investigate circuits comprised of a resistance $R$, an inductance $L$, and a capacitance $C$. Such circuits are usually called RLC circuits. The behavior of such circuits is novel and interesting, and is analogous to other systems in physics which exhibit resonance.

RLC circuit (transient response)
Consider the series RLC circuit shown in Fig. 1.
When the switch $S_1$ is closed, Kirchhoff's rule says that the sum of the potential drops across the circuit elements must be zero, i.e.,

$$V_L + V_R + V_C = 0 \quad (1)$$

The potential drop across the inductance $L$ is given by the equation

$$V_L = Lf_{\text{res}} \frac{dI}{dt} \quad (2)$$

The potential drop across the resistor $R$ is given by Ohm's law, i.e.,

$$V_R = RI \quad (3)$$

The potential drop across the capacitor $C$ is given by the equation

$$V_C = f_{\text{res}} qC \quad (4)$$

Substituting eqs. (2), (3), and (4) into eq. (1) gives

$$Lf_{\text{res}} \frac{dI}{dt} + RI + f_{\text{res}} qC = 0 \quad (5)$$

Differentiating once, we have

$$Lf_{\text{res}} \frac{d^2I}{dt^2} + Rf_{\text{res}} \frac{dI}{dt} + f_{\text{res}} C f_{\text{res}} q \frac{dq}{dt} = 0 \quad (6)$$

Recalling that the definition of current is

$$I = f_{\text{res}} \frac{dq}{dt} \quad (7)$$

we can substitute $I$ for $\frac{dq}{dt}$ in the last term of eq.(6) and get

$$Lf_{\text{res}} \frac{d^2I}{dt^2} + Rf_{\text{res}} \frac{dI}{dt} + f_{\text{res}} C f_{\text{res}} qI = 0 \quad (8)$$

Eq.(8) is analogous to the differential equation for a mechanical harmonic oscillator – a mass $m$ attached to a spring (with spring constant $k$) – which is subject to a damping force:

$$mf_{\text{res}} \frac{d^2x}{dt^2} + rf_{\text{res}} \frac{dx}{dt} + kx = 0 \quad (9)$$

Recall that for this mechanical system, if the mass were displaced from its equilibrium position and released, the mass would move sinusoidally, with the amplitude decreasing exponentially with time.

Since eqs.(8) and (9) are identical in form, one might expect that the current in an RLC circuit would behave similarly to the displacement in the mechanical system. Comparison of eqs.(8) and (9) suggest the following substitutions:
In other words, to complete the analogy between these particular systems, one must:

1. replace mass with inductance,
2. replace the damping constant with resistance,
3. replace the spring constant with the inverse of the capacitance, and
4. replace displacement with current.

The solution of eq.(8) is given in many texts, so we simply present it here:

\[ I = I_0 e^{-\frac{R}{2L} t} \cos(\omega t + \delta) \]  \hspace{1cm} (11)

It can also be shown that

\[ \omega = \sqrt{f_{\text{resonance}} \frac{1}{LC} - f_{\text{resonance}} R^2 4L^2} \]  \hspace{1cm} (12)

Note that in the special case of no damping, \( R = 0 \), eq.(12) reduces to

\[ \omega_0 = f_{\text{resonance}} \frac{1}{\sqrt{LC}} \]  \hspace{1cm} (13)

which is called the natural frequency without damping.

Putting the oscillation in terms of frequency \( f \) or period \( T \), we have

\[ f = f_{\text{resonance}} \frac{1}{T} = f_{\text{resonance}} \frac{2\pi}{T} = f_{\text{resonance}} \frac{12\pi}{\sqrt{f_{\text{resonance}} \frac{1}{LC} - f_{\text{resonance}} R^2 4L^2}} \]  \hspace{1cm} (14)

This solution applies only when \( \frac{1}{LC} \geq \frac{R^2}{4L^2} \).

A graph of eq. (11), i.e., the current as a function of time, is shown in Fig. 2. The “envelope” of the decay,

\[ I = I_0 e^{-\frac{R}{2L} t} \]  \hspace{1cm} (15)

Is also plotted. This function is of the form \( I = I_0 e^{-\lambda t} \) where the “time constant” is \( \tau = 2LR \).

![Figure 2: Damped harmonic motion, eq.(11)](image)

From these equations it is evident that the amplitude of the oscillation drops to \( \frac{1}{e} \) of initial value when \( e^{-\lambda t} = e^{-1} \) i.e., when \( t = \tau \):

\[ f_{\text{resonance}} R \tau 2L = 1 \]  \hspace{1cm} (16)
Driven RLC Circuits

Consider now an RLC circuit, as shown in Fig. 3, which is connected to a sine wave oscillator.

Kirchhoff’s rule in this case becomes

\[ V_L + V_R + V_C = V(t) \]  \hspace{1cm} (18)

Since the driving potential \( V(t) \) is harmonic, it can be represented by an equation of the form

\[ V(t) = V_0 \cos \omega t \]  \hspace{1cm} (19)

Substituting eqs. (2), (3), (4), and (19) into eq. (18), we obtain

\[ Lf_{\text{resonance}}dldt + Rl + f_{\text{resonance}}qlC = V_0 \cos \omega t \]  \hspace{1cm} (20)

Further, by definition,

\[ l = f_{\text{resonance}}dqdt \]  \hspace{1cm} (21)

Differentiating gives

\[ f_{\text{resonance}}dldt = f_{\text{resonance}}d^2qdt^2 \]  \hspace{1cm} (22)

Substituting eqs. (21) and (22) into eq. (20) gives

\[ Lf_{\text{resonance}}d^2qdt^2 + Rf_{\text{resonance}}dqdt + f_{\text{resonance}}1Cq = V_0 \cos \omega t \]  \hspace{1cm} (23)

Equation (23) is completely analogous to the equation of motion for the forced harmonic motion of a glider on an air track,

\[ mf_{\text{resonance}}d^2xdt^2 + rf_{\text{resonance}}dxdt + kx = F_0 \cos \omega t \]  \hspace{1cm} (24)

if we identify \( q \) with \( \dot{q} \) instead of with \( \dot{x} \). Thus, the response of the RLC circuit must be completely analogous to the forced motion of a glider mounted between two springs on a linear air track.

The solution of eq. (23) is

\[ \dot{q}(t) = f_{\text{resonance}}V_0/L\sqrt{(\omega^2 - \alpha^2)^2 + (f_{\text{resonance}}R\omega L)^2} \cos(\omega t + \delta) \]  \hspace{1cm} (25)

where

\[ \omega = \sqrt{f_{\text{resonance}}LC} \]  \hspace{1cm} (26)

and

\[ \tan \delta = f_{\text{resonance}}R\omega /La\omega^2 - \omega^2 \]  \hspace{1cm} (27)
It should be noted that eq.(25) is not the general solution: it is the steady state solution. Thus, just as in the mechanical system, the charge changes at the frequency of the driving voltage, and the amplitude and phase are functions of frequency. Generalizations of eqs.(25) and (27), the now familiar resonance forms, are plotted in Fig. 4. Here $A_R$ is the generalized amplitude at resonance.

![Figure 4: Resonant amplitude and phase vs. driving frequency, for Q=3](image)

A convenient description of the resonance curve makes use of the *quality factor*, $Q$. The $Q$ for the RLC circuit may be expressed in several equivalent forms,

$$Q = f_{\text{resonance}}f_0 \Delta f = f_{\text{resonance}}\omega_0 L R = f_{\text{resonance}}\omega_0 I_{\text{res}} e = f_{\text{resonance}}V_C(\text{res}) V_{\text{gen}}$$  \hspace{1cm} (28)

where $f_0$ and $\omega_0$ represent resonant values and $V_C(\text{res})$ is the potential across the capacitor at resonance. $V_C(\text{res})$ is a special case for the general form of $A_R$ in Fig. 4.

Referring to Fig. 2, $Q = \pi$ times the number of cycles which occur until the envelope drops to $1/e$ of its initial value.

**Experimental Procedure**

**Transient Response of RLC Circuit**

**Natural Frequency and Q of RLC Circuit**

Set up the circuit shown in Fig. 5.

Note that the inductance of the inductor is *10mH* and the capacitance of the capacitor is *0.01uf*. 
Set the resistance to 200 ohms on the R-box.

Set the function generator to a 1kHz square wave.

On the scope display the voltage across the capacitor. Zoom in on the point where the voltage switches from negative to positive. You will see the voltage oscillate sinusoidally before damping out, similar to the behavior of the current in figure 2.

The frequency of this oscillation is the natural frequency of the circuit, measure it on the scope and compare with the value calculated from equation 12. Remember that \( f = 2f \).

Measure the time \( t_{1/e} \) it takes the amplitude of the oscillation to fall by a factor of \( e^{-1} \). From equation 28 use your measured value for \( t_{1/e} \) and calculate the Q of the circuit as \( Q = \frac{\text{amp amplitude}}{50 \Omega} \). Calculate Q from equation 28 using the known values of L and R (don't forget the output impedance of the function generator).

### Forced Oscillation and Resonance of an RLC Circuit

Now connect the circuit shown in Fig. 6.

Set the function generator to a sine wave.

Measure the resonant frequency of the RLC circuit by finding the frequency at which the voltage across the capacitor is a maximum. Compare with the value of the resonant frequency determined above from the Transient Response of the circuit.
Q of the circuit.

To measure the Q of the circuit make a plot of the amplitude of the voltage across the capacitor as a function of the frequency of the sine wave. You should get a plot similar to the upper plot in figure 4.

Measure Q using equation 26.

\[ Q = \frac{f_{\text{resonance}} \text{ amplitude at resonance}}{\text{amplitude near zero frequency}} \] (26)

Note that you cannot actually measure the amplitude for a frequency of zero. What you will observe is that as you reduce the frequency of the sine wave the amplitude of the voltage across the capacitor will asymptotically approach a minimum value which is very close to the value at frequency of zero.

Report submission

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